Fractal and multifractal properties of exit times and Poincaré recurrences

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(Received 25 November 1996)

Systems with chaotic dynamics possess anomalous statistical properties, and their trajectories do not correspond to the Gaussian process. This property imposes description of such time characteristics as the distribution of exit times or Poincaré recurrences by introducing a (multi-) fractal time scale in order to satisfy the observed powerlike tails of the distributions. We introduce a corresponding phase-space-time partitioning and spectral function for dimensions, and make a connection between dimensions and transport exponent that defines the anomalous (''strange'') kinetics. [S1063-651X(97)15405-8]

PACS number(s): 05.45.+b

I. INTRODUCTION

After a set of pioneering publications [1–4], fractal and multifractal analyses of the dynamical systems became routine methods for the diagnostics of chaos. A rigorous definition of fractals, or more accurately, a dimensionlike characteristics, was considered in [5], where the so-called Pesin's dimension was introduced as a generalization of the Caratéodory dimension [6]. Later it was shown in [7,8] that the intuitively introduced multifractal description of dimension properties and the famous $f(\alpha)$ spectrum for dimensions can also be defined in a rigorous way.

The peculiarity of (multi-) fractal analysis of dynamical systems, as compared to static geometric objects, is that there exists a theoretical possibility to obtain all necessary information from a single trajectory. In fact, one would like to explain the (multi-) fractal properties of a trajectory that fills the phase space and has a very complicated distribution of the occupation time for different phase space domains. There are many fractal objects in the phase space of a chaotic system. Let us mention only such objects as cantori and islands-around-islands (see review [9]). Recently the kinetics of Hamiltonian chaotic systems has been considered to be fractal in space and time simultaneously. The popular notion of fractals [10] can be extended, and a fractal time introduced for the random processes of Lévy-type applied to different kinds of systems [11–13].

For Hamiltonian systems with chaotic dynamics, the motion is not ergodic in the full phase space, and one needs to extract a (multi) fractal set of islands to obtain a domain with ergodic trajectories. The islands are a singular part of the phase space. The behavior of the trajectories near an island boundary layer was studied in [14–18] as a fractal object which imposes powerwise distributions in the large time asymptotics of chaotic kinetics. More specifically, the island boundary is sticky, the subisland boundary is more sticky, and so on. As a result of the situation described, fractal space-time properties of the trajectories were considered. We encounter a situation in which (i) fractal properties exist simultaneously in space and time, and (ii) the multiplicity of the resonance sets that generate islands is adequate to a multifractal construction of the trajectories rather than to a fractal one.

A description of the multifractal time and the corresponding spectral function of dimensions is the subject of this work. For a general situation of chaotic dynamics, the fractal time cannot be introduced without considering a space structure. For that reason, the space-time coupling is nontrivial, and so is the spectral function of the fractal indices. In Sec. II we briefly describe the necessary characteristics of the Hamiltonian chaotic dynamics, with a particular emphasis on the Poincaré recurrences and exit time distributions. In Sec. III we consider fractal space-time structures, and in Sec. IV multifractal ones.

II. SCALING PROPERTIES OF TRAJECTORIES AND ASYMPTOTIC DYNAMICS

Distribution function for trajectories in phase space can be fairly uniform, as happened, for example, in the ''Arnold cat'' system. Nevertheless, a typical Hamiltonian system has a rich set of islands in phase space, with a regular dynamics inside the islands and with narrow stochastic layers isolated from the main stochastic sea domain. As an example, one can consider the web map

$$\overline{u}=v, \quad \overline{v}=-u-K\sin v \tag{2.1}$$

or the standard map

$$\overline{p}=p-K\sin x, \quad \overline{x}=x+\overline{p}, \tag{2.2}$$

with a fairly well known island structure which will be discussed more below. The dynamics near the islands boundary is singular due to the phenomena of stickiness, and it can dominate in the large time asymptotics. This circumstance influences almost all important probability distributions such as the distribution of distances, exit times, recurrences, moments, etc. The main feature of all such distributions is that they do not correspond to either Gaussian or Poissonian (or similar) processes with all finite moments. This is due to the presence of powerlike tails in the asymptotical limits of large space-time scales.

Being more specific, one can say that power-wise tailed distributions are a consequence of a (multi-) fractal singular scattering zone near the island boundaries. More precisely, there are different sets of islands with different asymptotics (different powers of distribution tails) and different scales for where and when the asymptotics work. Different intermediate asymptotics is a crucial characteristic of the anomalous transport, as was mentioned in the problem of advection [19], and the problem of charged particle motion in an electromagnetic field [20]. This is the basis for introducing a multifractal description of some distributions in chaotic dynamics.

As an example, let us consider an element $\Delta\Gamma$ of the finite phase volume Γ in which the full area of stochastic motion (stochastic sea) is located. The system is Hamiltonian and the dynamics is area preserving. Introduce a set of time instants $\{t_j\}$ when a trajectory crosses the boundary of $\Delta\Gamma$ on the way from inside $\Delta\Gamma$ to outside. The intervals

$$\tau_j=\{t_{j+1}-t_j\}, \quad j=0,1,\ldots \tag{2.3}$$

are Poincaré cycles. Their distribution $P(\tau,\Delta\Gamma)$ can be normalized as

$$\frac{1}{\Delta\Gamma}\int_0^\infty P(\tau;\Delta\Gamma)d\tau=1, \tag{2.4}$$

and the limit

$$P(\tau)=\lim_{\Delta\Gamma\to 0}P(\tau;\Delta\Gamma)/\Delta\Gamma \tag{2.5}$$

exists if there exists ergodicity [21] and phase space compactness. Moreover, if the measure of the chaotic orbits is nonzero, then

$$\langle\tau\rangle\equiv\tau_0<\infty, \tag{2.6}$$

i.e., the mean recurrence time is finite [21] (see also [22]). Using the results of [21] and the expansion form [22] it is possible to show [23] that in the asymptotic formula

$$P(\tau)\sim\tau^{-\gamma}, \quad \tau\to\infty \tag{2.7}$$

there exists a restriction

$$\gamma>2. \tag{2.8}$$

The chaotic dynamics can be considered as a normal one if the Poissonian distribution

$$P(\tau)=\frac{1}{\langle\tau\rangle}\exp(-\tau/\langle\tau\rangle) \tag{2.9}$$

or any similar distribution with all finite moments is valid. Such a situation was described in [24–27].

It was mentioned in [26] that the anomalous behavior (2.7) correlates with the anomalous transport situation when the moments of the displacement $R(t)$ satisfy the asymptotic equation

$$\langle R^{2n}(t)\rangle\sim t^{\mu n}, \quad t\to\infty, \tag{2.10}$$

with the transport exponent $\mu\neq 1$. It is worthwhile to mention the results for the anomalous statistical properties and their correlation with either the Poincaré recurrences or escape time distributions for the standard map [28–31] and for the Lorenz gas (Sinai billiard) with infinite horizon [32–37]. In [17,18] and [38], different approaches were proposed to consider Poincaré recurrences by introducing a dimension-like characteristic or, in other words, by using a fractal description of the recurrence times as a random set (2.3). Particularly, it was found in [18] that there exists a connection between the recurrence times exponent γ in Eq. (2.7) and the transport exponent μ in Eq. (2.10), namely,

$$\gamma=2+\mu, \tag{2.11}$$

which is inferred by a specific phase space topology of maps (2.1) and (2.2). Connection (2.11) was confirmed by high accuracy simulation in [17,18,39]. These results encouraged us to consider the fractal structure of islands in the phase space as a seed of fractal and multifractal structures of recurrence times. We can add to the consideration distribution of the escape times from different boundary layer zones [17,18] by introducing a time delay distribution. Let $\psi(t;\Delta\Gamma)$ be the probability density to escape from $\Delta\Gamma$ during the time interval $(t,t+dt)$. Then the probability to escape from $\Delta\Gamma$ for the time not less than t is

$$P_e(t;\Delta\Gamma)=\int_0^t d\tau\ \psi(\tau;\Delta\Gamma). \tag{2.12}$$

The corresponding survivial probability for the time t is

$$\Psi(t;\Delta\Gamma)=1-P_e(t;\Delta\Gamma)=1-\int_0^t d\tau\ \psi(\tau;\Delta\Gamma), \tag{2.13}$$

with the normalization condition

$$P_e(t\to\infty;\Delta\Gamma)=1. \tag{2.14}$$

The mean time of trapping to the domain $\Delta\Gamma$ is

$$t_s(\Delta\Gamma)=\int_0^\infty d\tau\ \tau\psi(\tau;\Delta\Gamma). \tag{2.15}$$

In the case where there is only one ''leading'' exponent in Eq. (2.7), we can assume that the same has happened with the escape distribution function $\psi(\tau;\Delta\Gamma)$ i.e.,

$$\psi(\tau;\Delta\Gamma)\sim\tau^{-\gamma}, \tag{2.16}$$

and then, in correspondence with Eq. (2.11) and the condition $\mu>0$, we obtain restriction (2.8). The one-exponent (or fractal) situation is not sufficient to describe a typical situation, and in the following sections we reconsider it using a more solid basis.

III. FRACTAL SPACE-TIME PARTITIONING

Consider the space-time partitioning that was introduced in [15] (see also [18]) and resembles the Sierpinsky carpet (Fig. 1). Let the central square be an island of zero generation. Surround the island by an annulus which represents the

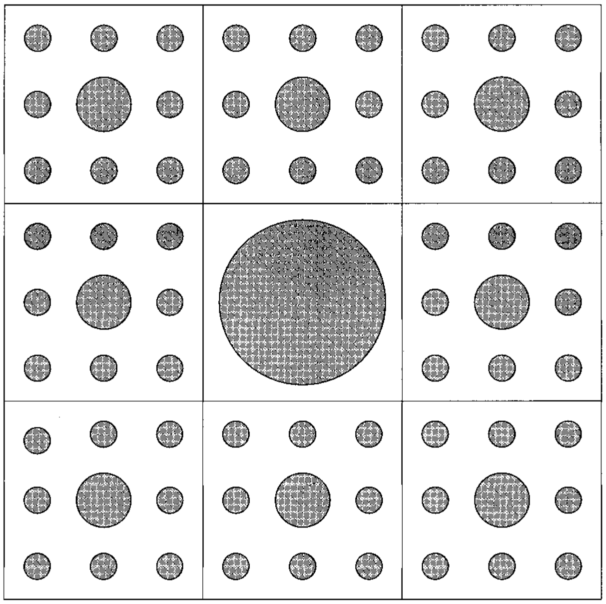

FIG. 1. Scheme of the phase space for islands surrounded by subislands.

boundary island layer. It consists of g_1 ($g_1=8$ in Fig. 1) subislands of the first generation (dashed small islands in Fig. 1). We can partition the annulus by g_1 domains, so that each of them includes exactly one island of the first generation; then we surround each of the first generation islands by an annulus of the second generation and repeat the process. On the nth step the structure can be described by a ''word''

$$w_n = w(g_1, g_2, \ldots, g_n). \tag{3.1}$$

The full number of islands on the nth step is

$$N_n = g_1, \ldots, g_n, \tag{3.2}$$

and any island from the nth generation can be labeled by

$$u_i^{(n)} = u(i_1, i_2, \ldots, i_n), \quad 1 \leqslant i_j \leqslant g_j, \ \forall j. \tag{3.3}$$

Let us now introduce the time that a particle spends in the boundary layer of an island. This time,

$$T_i^{(n)} = T(u_i^{(n)}), \tag{3.4}$$

carries all information of the nth generation islands (3.1)–(3.3). By introducing a residence time for each island boundary layer, we have a situation comparing to the plain Sierpinsky carpet or plain fractal situation because of the nontriviality of the space-time coupling. In fact, we are attaching an additional parameter responsible for the temporal behavior to the simple geometric construction which is similar to a Cantor set.

A simplified situation corresponds to the exact self-similarity of the construction described above, i.e.,

$$S_i^{(n)} = S^{(n)} = \lambda_S^n S^{(0)}, \quad \forall i,$$

$$T_i^{(n)} = T^{(n)} = \lambda_T^n T^{(0)}, \quad \forall i, \tag{3.5}$$

where $S_i^{(n)}$ is area of an island $u_i^{(n)}$ and $T_i^{(n)}$ is introduced in Eq. (3.4). Expressions in Eq. (3.5) correspond to equal areas and residence times for all islands of the same generation. Two scaling parameters λ_S and λ_T represent the existence of the exact self-similarity in space and time correspondingly. Precisely such a situation was described in [15–18] for maps (2.1) and (2.2), with

$$\lambda_S < 1, \quad \lambda_T > 1. \tag{3.6}$$

In addition to Eq. (3.5), there is a self-similarity in the islands' proliferation, i.e.,

$$g_n = \lambda_g^n g_0, \quad \lambda_g \geqslant 3 \tag{3.7}$$

It follows from Eqs. (3.2) and (3.7) that

$$N_n = \lambda_g^n g_0 = \lambda_g^n \tag{3.8}$$

if we start from the only island ($g_0 = 1$). It is useful to introduce a ''residence frequency''

$$\omega_i^{(n)} = 1/T_i^{(n)} \tag{3.9}$$

with the self-similarity property

$$\omega_i^{(n)} = \omega^{(n)} = \lambda_T^{-n} \omega^{(0)}. \tag{3.10}$$

Consider now a partitioning which corresponds to conditions (3.1)–(3.4) with the simplification (3.7). The nth level of the partitioning corresponds to the nth level of the islands' hierarchy, i.e., each space bin has an area $S_i^{(n)}$ and a cojoint residence time $T_i^{(n)}$ [both are defined in Eq. (3.5)] whose sizes do not depend on i. The elementary probability for a bin to spend time $T_i^{(n)}$ in the domain $S_i^{(n)}$ can be presented in a simple form

$$P_i^{(n)} \equiv P_{i_1, i_2, \ldots, i_n} = C_n \omega_i^{(n)} S_i^{(n)}, \quad \forall 1 \leqslant i_j \leqslant g, \tag{3.11}$$

where C_n is a normalization constant. Let us call $P_i^{(n)}$ an elementary bin probability. Using Eqs. (3.5), (3.6), and (3.9), we can rewrite Eq. (3.11) as

$$P_i^{(n)} = C_n (\lambda_S / \lambda_T)^n, \quad \forall i. \tag{3.12}$$

With expression (3.12), we can consider different sums and partition functions. As an example, consider the sum

$$\sum_{i_1, \ldots, i_n} P_i^{(n)} = C_n \sum_{i_1, \ldots, i_n} \exp[-n(|\ln \lambda_S| + \ln \lambda_T)]$$

$$= \sum_{i_1, \ldots, i_n} \exp[-n(|\ln \lambda_S| + \ln \lambda_T) + n \psi_n] = 1, \tag{3.13}$$

where the ''free energy'' density

$$\psi_n = \frac{1}{n} \ln C_n. \tag{3.14}$$

is introduced. The number of terms in Eq. (3.13) follows from Eq. (3.8), and therefore in the limit $n \to \infty$ we obtain

$$\lim_{n\to\infty} \psi_n = \psi = |\ln\lambda_S| + \ln\lambda_T - \ln\lambda_g . \tag{3.15}$$

A more precise formulation of result (3.15) is that the sum in Eq. (3.13) diverges if $\psi_n > \psi$, converges to zero if $\psi_n < \psi$, and converges to 1 if $\psi_n = \psi$.

IV. MULTIFRACTAL SPACE-TIME AND ITS DIMENSION SPECTRUM

It was mentioned in Secs. I and II that chaotic dynamical systems with rich sets of islands have a multifractal structure rather than fractal space-time structure. Our purpose now is to introduce a spectral function of dimensions in analogy to [1–4].

Following a usual method of statistical mechanics, let us introduce a partition function in the form

$$Z_{\mathrm{discr}}^{(n)}\{\lambda_T,\lambda_S;q\} = \sum_{i_1,i_2,\dots,i_n} (\omega_i^{(n)} S_i^{(n)})^{\gamma q}. \tag{4.1}$$

Here we use the space-time bin probability $\omega_i^{(n)} S_i^{(n)}$ introduced in Eq. (3.11). Considering a multiscaling situation, we assume that a real elementary probability to occupy a bin should have the same scaling dependence as in Eq. (3.11) up to a power of γ, and that there are different values of γ in the sum. With an exponent q we can consider different moments of the elementary bin probability. In particular, for $q=0$, we simply obtain a number of bins. Let us replace summation by integration and write

$$Z^{(n)}\{\lambda_T,\lambda_S;q\} = \int d\gamma \rho(\gamma) [\omega^{(n)} S^{(n)}]^{-f(\gamma)+\gamma q}, \tag{4.2}$$

where the density of the space-time bins is introduced:

$$dN^{(n)}(\gamma) = d\gamma \rho(\gamma) [\omega^{(n)} S^{(n)}]^{-f(\gamma)}. \tag{4.3}$$

The function $f(\gamma)$ is a spectral function of the space-time dimensionlike characteristics, or simply, dimensions. The distribution density $\rho(\gamma)$ is a slow function of γ. To be more accurate, we should assume that the bin probability $\omega^{(n)} S^{(n)}$ also depends on γ because for different island sets the bins have different structures. Nevertheless, the dependence of $\omega^{(n)} S^{(n)}$ on γ is slow in comparison to the exponential low in Eq. (4.3).

Using Eqs. (3.5) and (3.10) transforms Eq. (4.2) into

$$Z^{(n)}\{\lambda_T,\lambda_S;q\} = \int d\gamma \rho(\gamma) \exp\{-n[\gamma q - f(\gamma)] \times (|\ln \lambda_S| + \ln \lambda_T)\}, \tag{4.4}$$

where λ_S and λ_T are slow functions of γ. For $n\to\infty$ the standard steepest descent procedure gives

$$Z^{(n)}\{\lambda_T,\lambda_S;q\} \sim \exp\{-n[\gamma_0 q - f(\gamma_0)](|\ln\lambda_S| + \ln\lambda_T)\}, \tag{4.5}$$

with the equation to determine $\gamma_0 = \gamma_0(q,\lambda_S,\lambda_T)$,

$$q = f'(\gamma_0). \tag{4.6}$$

From another side recall that, for $q=0$, expression (4.1) defines $Z_{\mathrm{discr}}^{(n)}\{\lambda_T,\lambda_S;0\}$ as a number of bins. For the one-scale situation we have this number from Eq. (3.8) as λ_g^n. For the multifractal case we can write a power of λ_g^n by introducing a generalized dimension $\mathcal{D}_q$, in analogy to [39,1–4]

$$Z^{(n)}\{\lambda_S,\lambda_T;q\} \sim \lambda_g^{-n(q-1)D_q} \sim \exp\{-n(\ln \lambda_g)(q-1)D_q\}. \tag{4.7}$$

The scaling parameter λ_g defines a coefficient of proliferation of space-time bins, and we can consider

$$\lambda_g = \lambda_g(\lambda_T,\lambda_S), \tag{4.8}$$

i.e., that we consider dynamical systems with only two independent scaling parameters.

A comparison of Eqs. (4.7) and (4.5) gives

$$(q-1)D_q \ln\lambda_g = [\gamma_0 q - f(\gamma_0)](\ln\lambda_T + |\ln\lambda_S|). \tag{4.9}$$

For some limit cases, λ_g in Eq. (4.8) should satisfy the conditions

$$\lambda_g = \begin{cases} \lambda_S^{-1} & \text{if } \lambda_T = 1 \\ \lambda_T & \text{if } \lambda_S = 1. \end{cases} \tag{4.10}$$

For a general situation, $\lambda_g,\lambda_T,\lambda_S \neq 1$, and we can rewrite Eq. (4.9) in the final form

$$(q-1)D_q = \frac{\ln\lambda_T}{\ln\lambda_g} (1+\mu)[\gamma_0 q - f(\gamma_0)], \tag{4.11}$$

where the parameter

$$\mu = |\ln\lambda_S| / \ln\lambda_T \tag{4.12}$$

is introduced in [15–18], and called the transport exponent (see (2.10)).

It follows from Eq. (4.11) that, for $q=0$,

$$D_0 = \frac{\ln\lambda_T}{\ln\lambda_g} (1+\mu) f(\gamma_0), \tag{4.13}$$

i.e., there is now no simple connection between the dimension D_0 and the spectral function. The regular formula

$$D_0 = f(\gamma_0) \tag{4.14}$$

appears only in the cases (4.10) when one has a multifractal structure only in space or time. For $q=1$, using Eqs. (4.6) and (4.11), we obtain

$$D_1 = \gamma_0(1) \frac{\ln\lambda_T}{\ln\lambda_g} (1+\mu), \tag{4.15}$$

where the value $\gamma_0(1) = \gamma_0(q=1)$ can be obtained from Eq. (4.6): $f'[\gamma_0(q=1)] = 1$.

In Eq. (4.9) we expressed the generalized dimension D_q through the spectral function $f(\gamma)$ as in [1–4]. Nevertheless formulas (4.9), (4.11), (4.13), and (4.15) show that the knowledge of the spectral function is not sufficient for a typical dynamical system, and some additional information is necessary about the system's structure in space and time.

A simplification can be used for the webmap (2.1), where $\lambda_g = \lambda_T$ [17,18] but it can be a case when $\lambda_g \neq \lambda_T$ for the standard map (2.2) [18,40].

V. CRITICAL EXPONENT FOR THE POINCARÉ RECURRENCES

Consider again the partitioning introduced in Sec. III (see Fig. 1) and recurrences of escapes from a boundary island layer of the nth generation. When normalized to the unit of the probability (3.11) to occupy a bin by a particle, the corresponding ''number of states'' can be written in the form

$$Z_r^{(n)} = \sum_{i_1,\ldots,i_n} \frac{1}{S_i^{(n)}\omega_i^{(n)}} = \sum_{i_1,\ldots,i_n} (\lambda_T/\lambda_S)^n. \tag{5.1}$$

Instead of Eq. (5.1), consider a more general expression

$$Z_r^{(n)}(q) = \sum_{i_1,\ldots,i_n} \frac{1}{S_i^{(n)}[\omega_i^{(n)}]^q} = \sum_{i_1,\ldots,i_n} \lambda_T^{nq}/\lambda_S^{-n}. \tag{5.2}$$

Using Eq. (3.8), we have the estimation

$$Z_r^{(n)}(q) \sim \exp\{n(q \ln \lambda_T + |\ln\lambda_S| + \ln\lambda_g)\}. \tag{5.3}$$

The expression can be simplified if we consider the case $\lambda_g = \lambda_T$ [17,18] when the proliferation coefficient for the number of islands coincides with the coefficient of the increase in the circulating period around the islands. Then

$$Z_r^{(n)}(q) \sim \exp\{n[(q+1)\ln\lambda_T + |\ln\lambda_S|]\}. \tag{5.4}$$

This expression is finite if

$$q \leqslant q_c = -(|\ln\lambda_S| + \ln\lambda_T)/\ln\lambda_T = -(1+\mu). \tag{5.5}$$

The obtained result has a remarkable interpretation.

From definition (5.2) we can consider $Z_r^{(n)}(q)$ as time moments of order q for the escape or recurrence sum of the states $Z_r^{(n)}$. It is finite only if $|q| \leqslant |q_c|$. That means that the probability density for recurrences [see Eq. (2.7)] should possess the asymptotics

$$P(t) \sim t^{-1+q_c} = t^{-1-(1+\mu)} = t^{-2-\mu} \tag{5.6}$$

in order to have finite moments of order $q > 0$. Result (5.6) coincides with the expressions (2.11) and (2.7) derived in [18] from different consideration.

VI. TWO-ISLAND-SET MODEL

Consider a case when the hierarchy of islands can be represented by two independent sets of islands with two corresponding pairs of scaling coefficients: $(\lambda_S^{(1)}, \lambda_T^{(1)})$ and $(\lambda_S^{(2)}, \lambda_T^{(2)})$. We consider a simplified version

$$\lambda_T^{(1)} = g^{(n)}, \quad \lambda_T^{(2)} = g^{(2)}, \tag{6.1}$$

where $g^{(1)}$ and $g^{(2)}$ represent the proliferation of island numbers for the first and second sets, correspondingly. In analogy to Eq. (3.3) let us label an island of the nth generation as follows

$$u_i^{(n)} = u(i_1, i_2, \ldots, i_n), \quad (1 \leqslant i_j \leqslant g^{(1)} + g^{(2)}, \forall j) \tag{6.2}$$

and

$$\lambda_T(i_j) = \begin{cases} \lambda_T^{(1)}, & 1 \leqslant i_j \leqslant g^{(1)} \\ \lambda_T^{(2)}, & g^{(1)}+1 \leqslant i_j \leqslant g^{(1)} + g^{(2)}, \end{cases} \tag{6.3}$$

$$\lambda_S(i_j) = \begin{cases} \lambda_S^{(1)}, & 1 \leqslant i_j \leqslant g^{(1)} \\ \lambda_S^{(2)}, & g^{(1)}+1 \leqslant i_j \leqslant g^{(1)} + g^2. \end{cases} \tag{6.4}$$

Continuing the analogy to the simple-set island model, let us write an expression similar to Eq. (5.2) for the number of states' moments:

$$Z_r^{(n)}(q) = \sum_{i_1, i_2, \ldots, i_n} \prod_{j=1}^{n} \lambda_T^q(i_j)/\lambda_S(i_j). \tag{6.5}$$

We can use the expression

$$Z_r^{(n)}(q,\gamma) = \sum_{i_1,\ldots,i_n} \exp\left(\sum_{j=1}^{n} q \ln\lambda_T(i_j) - \ln\lambda_T(i_j) \right) e^{-\gamma n} \tag{6.6}$$

to define the dimension spectral function $\gamma = \gamma_c(q)$ as a critical value of γ for which the sum (6.6) converges. This value can be expressed as

$$\gamma_c(q) = \ln\nu(q), \tag{6.7}$$

where $\nu(q)$ is the maximal eigenvalue of the matrix $\mathbf{A} = \mathbf{C} \cdot \mathbf{C}(q)$ [41] with

$$\mathbf{C}(q) = \mathrm{diag}(\underbrace{(\lambda_T^{(1)})^q/\lambda_S^{(1)}, \ldots, (\lambda_T^{(1)})^q/\lambda_S^{(1)}}, \underbrace{(\lambda_T^{(2)})^q/\lambda_S^{(2)}, \ldots, (\lambda_T^{(2)})^q/\lambda_S^{(2)}})$$

$$\mathbf{A} = \left| \begin{matrix} (\lambda_T^{(1)})^q/\lambda_S^{(1)}, \ldots, (\lambda_T^{(1)})^q/\lambda_S^{(1)}, (\lambda_T^{(2)})^q/\lambda_S^{(2)}, \ldots, (\lambda_T^{(2)})^q/\lambda_S^{(2)} \\ (m_1+m_2) \\ (\lambda_T^{(1)})^q/\lambda_S^{(1)}, \ldots, (\lambda_T^{(1)})^q/\lambda_S^{(1)}, (\lambda_T^{(2)})^q/\lambda_S^{(2)}, \ldots, (\lambda_T^{(2)})^q/\lambda_S^{(2)} \end{matrix} \right|,$$

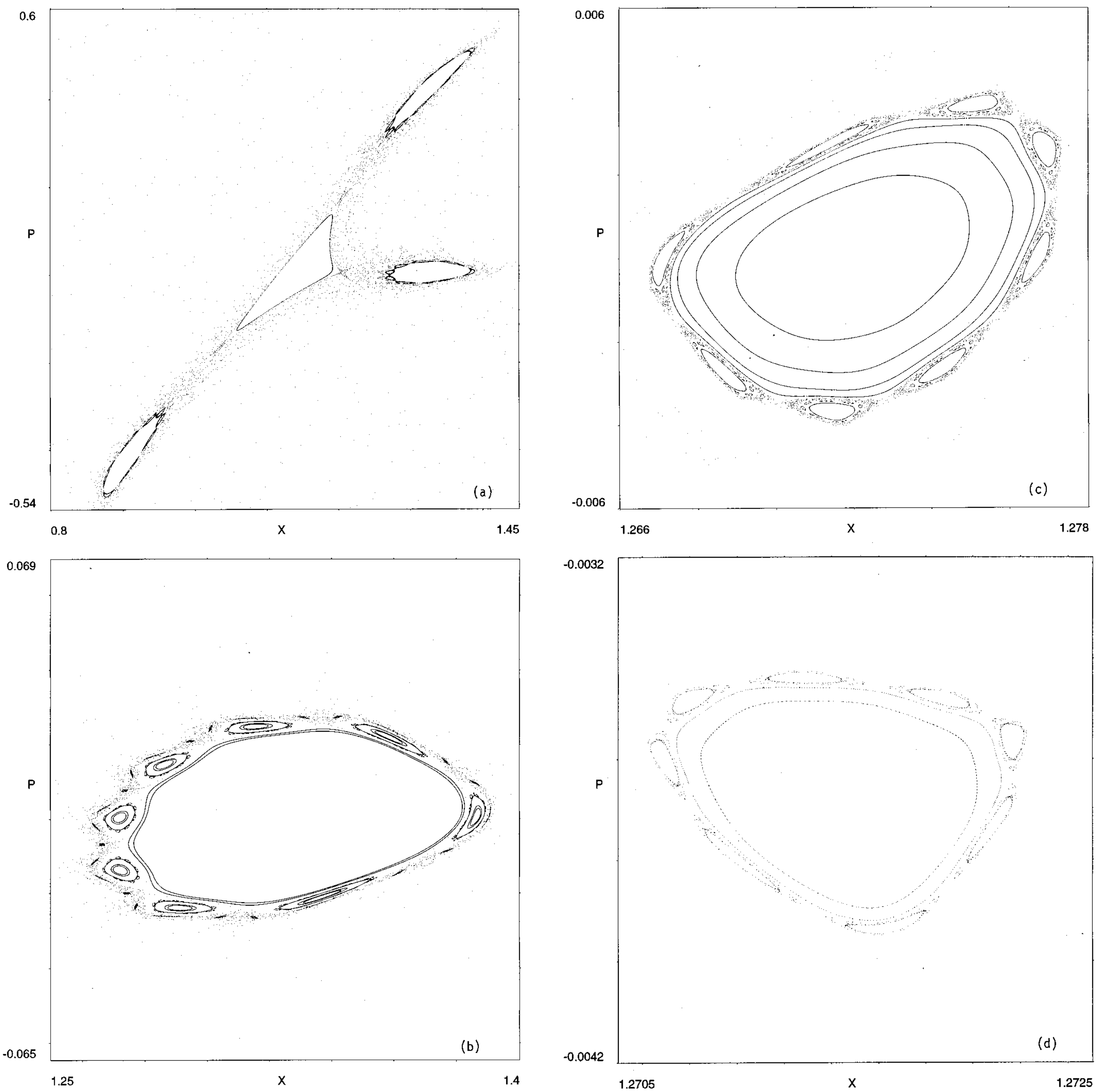

FIG. 2. Island hierarchies for the standard map with parameter value (7.1): (a) The main island and first generation of a three-island chain. (b) Magnification of the right island in (a). (c) Magnification of the left island in (b). (d) Magnification of the bottom island in (c).

and **B** if the $(m_1+m_2)\times(m_1+m_2)$ matrix consists of only units. The straightforward calculation gives

$$\nu(q)=(\lambda_T^{(1)})^{q+1}/\lambda_S^{(1)}+(\lambda_T^{(2)})^{q+1}/\lambda_S^{(2)}. \tag{6.8}$$

Hence, from Eqs. (6.7) and (6.8),

$$\gamma_c(q)=\ln[\lambda_T^{(1)})^{q+1}/\lambda_S^{(1)}+(\lambda_T^{(2)})^{q+1}/\lambda_S^{(2)}]. \tag{6.9}$$

Particularly for the cases $\lambda_T^{(1)}=\lambda_T^{(2)}=\lambda_T$ and $\lambda_S^{(1)}=\lambda_S^{(2)}=2\lambda_S$, we arrive at the result

$$\gamma_c(q)=|\ln\lambda_S|+(q+1)\ln\lambda_T=q\ \ln\lambda_T+(1+\mu)\ln\lambda_T. \tag{6.10}$$

From Eq. (6.10) one can find a critical value q_c from the condition $\gamma_c(q_c)=0$, i.e., $q_c=-(1+\mu)$, in correspondence with Eq. (5.5).

For the cases of slightly different scaling parameters for two sets of islands,

$$\lambda_S^{(1)}=2\lambda_S,\quad \lambda_S^{(2)}=2(\lambda_S+\delta\lambda_S)$$

$$\lambda_T^{(1)}=\lambda_T,\quad \lambda_T^{(2)}=\lambda_T+\delta\lambda_T \tag{6.11}$$

and $\delta\lambda_S \ll \lambda_S$, $\delta\lambda_T \ll \lambda_T$, it is easy to obtain, from Eq. (6.9),

$$\gamma(q) \approx -\ln(\lambda_S + \delta\lambda_S/2) + (q+1)\ln(\lambda_T + \delta\lambda_T/2). \quad (6.12)$$

As the critical value of q_c satisfies the condition $\gamma_c(q_c)=0$, we can obtain q_c using Eqs. (6.9) or (6.8) with the condition

$$(\lambda_T^{(1)})^{q+1}/\lambda_S^{(1)} + (\lambda_T^{(2)})^{q+1}/\lambda_S^{(2)} = 1, \quad (6.13)$$

or, for the case Eq. (6.11),

$$q_c \approx -1 - |\ln(\lambda_S + \delta\lambda_S/2)|/\ln(\lambda_T + \delta\lambda_T). \quad (6.14)$$

Equation (6.13) looks similar to the Moran equation for the Hausdorff dimension of some fractal sets [42], but with additional weights $\lambda_T^{(1)}/\lambda_S^{(1)}$ and $\lambda_T^{(2)}/\lambda_S^{(2)}$.

In fact, formula (6.13) can be easily generalized to obtain critical exponents q_c for the multifractal situation

$$\sum_j (\lambda_S^{(j)})^{q+1}/\lambda_T^{(j)} = 1. \quad (6.15)$$

The best way to obtain Eq. (6.15) is by induction using the additive form (6.13). The corresponding generalization of Eq. (6.9) will be

$$\gamma_c(q) = \ln \sum_j (\lambda_S^{(j)})^{q+1}/\lambda_T^{(j)}. \quad (6.16)$$

We should remember that the multiple island set considered here is only a possible situation. Different variants of how to build a multifractal for a multi-island set depends on the resonance structures of the system.

VII. NUMERICAL ANALYSIS

It is well known that the phase space of a typical Hamiltonian system with chaotic dynamics is a ''zoo'' of islands. They have a different nature, a different evolution with respect to change of parameters, and a different structure. Actually, we never have special kinds of islands, but always an infinite set of different island species. To demonstrate the existence of at least two scaling parameters (λ_S, λ_T) one should find a special value(s) of the control parameter and a specific set of islands with stickiness and proliferation parameter λ_g. Such an example, with some variations of $(\lambda_S, \lambda_T, \lambda_g)$, was demonstrated in [14] for an advection problem in the hexagonal helical flow. It is important to see an exact (or almost exact) self-similarity without variations of $(\lambda_S, \lambda_T, \lambda_g)$ along the island set in order to clarify the existence of the corresponding fractal structure of the dynamics in the islands' boundary layers. Then, by a small variation of a parameter, one can create a multifractal situation. In particular, the existence of the fractal situation was demonstrated in [17,18] for the web map, and in [18,40] for the standard map. Below we provide a similar example for the standard map (2.2).

In Fig. 2(a), we see the structure of four islands with the central one- and three-island resonance sets around the central island. The three-island set occurred as a result of a bifurcation when the three islands separated from the central one after the parameter K exceeded some critical value. If we continue to increase K, a similar structure of subislands occurs for the three satellite islands of the first generation. In Fig. 2(b), we show a magnification of the right island of the first generation. There is eight-island chain around it. One can find such a value of K, namely,

$$K_8 = 6.908\,745; \quad (7.1)$$

thus the proliferation number of islands is constant: $\lambda_g = 8$. Figures 2(c) and 2(d) show the next two generations of the eight-island chain. We can continue this demonstration at least for three more generations for the same value of K_8.

With the structure obtained, we can calculate the areas and the last invariant curve periods for islands of different generations. These data are presented below:

$$\ln\lambda_T \approx 2.1, \quad |\ln\lambda_S| \approx 2.5. \quad (7.2)$$

Values (7.2) can be used to find the parameter μ in Eq. (4.12)

$$\mu = |\ln\lambda_S|/\ln\lambda_T = 1.2, \quad (7.3)$$

i.e., a so-called transport exponent [see Eq. (2.10)]. Now the value μ can be compared with what we can obtain indepen-

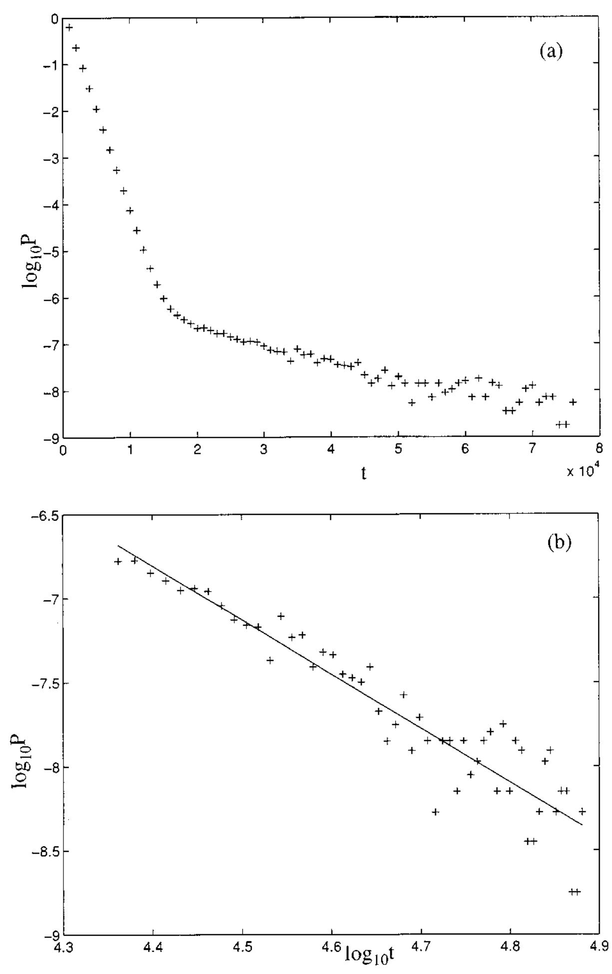

FIG. 3. Distribution function of the Poincaré cycles for the standard map case indicated in Fig. 2: (a) $\log_{10} P$ vs t plot; (b) $\log_{10} P$ vs $\log_{10} t$ plot for the tail.

dently by analyzing the Poincaré cycles distribution and applying formulas (2.11). Numerical results were obtained for the same parameter K Eq. (7.1), to obtain the Poincaré cycles distribution. They were collected from 5.5×10^5 initial conditions each run 10^6 iterations. Figure 3(a) displays the Poissonian law for the recurrence time $<1.5\times10^4$ steps, and then a crossover to a long powerlike tail. The analysis of the tail [Fig. 3(b)] gives its slope $\gamma=3.2\pm0.2$. This result, in combination with Eq. (7.3), shows that the formulas (2.11) works perfectly. From another side, due to Eq. (4.12),

$$\gamma=2+\mu=2+|\ln\lambda_S|/\ln\lambda_T, \tag{7.4}$$

and this means that computations confirm nontrivial dependence of the exponent γ of the Poincaré cycles distribution on the space and time fractal properties simultaneously.

VIII. CONCLUSION

In dealing with Hamiltonian chaotic dynamics, we met a situation in which the (multi-) fractal properties of motion are revealed in both phase space and time. This happened because of the presence of highly complex islands structures and the stickiness of their boundaries. At the moment, we are not familiar with a possibility to extend such a property of chaotic dynamics to dissipative systems or to those with more degrees of freedom. Nevertheless one can expect that the same kind of space-time (multi-) fractality is a typical property of the dynamical chaos that can be applied to more sophisticated problems such as turbulent flow.

With some simplification one can say that our generalization considers a fractal support characterized by a multidimensional space of variables λ_g, λ_S, and λ_T. It is possible to add here the scaling parameter for Lyapunov exponents λ_σ. It seems that, for simplified models, such as the standard map or web map, two scaling parameters, say λ_S and λ_T, are sufficient, because λ_g can be expressed through λ_T, and λ_σ should coincide with λ_T. This restriction was discussed in [14,15] in more details. For some cases, i.e., special sets of islands, we even expect the existence of a connection between λ_T and λ_S. Nevertheless such a connection cannot be imagined for a general situation, and a scheme for the proliferation of islands can be too complicated and nonuniversial to allow reducing the structure of the fractal support of chaotic dynamics to a single scaling parameter. It seems reasonable that one scaling parameter case occurs only as an approximation when all others are close to 1, and that the fractal situation occurs only as a narrow multifractal spectrum case.

A similar consideration can be extended to the spectral function of dimensions. How many independent spectral functions should define the multifractal description of chaotic dynamics, and particularly the exit time and Poincaré recurrences distribution? It is known that Lyapunov exponents characterize an intrinsic property of chaotic dynamics related to the local instability increment. It comes very naturally to introduce a spectral dimension function for the Lyapunov exponents [43,44]. In fact, it can be sufficient to describe the fractal properties of the system if no more than one scaling parameter or scaling spectral function is necessary for the description. Our generalization is introduced for the cases when only one function of two scaling parameters is necessary, and this is precisely the case for the distribution of Poincaré recurrences when the characteristic exponent (2.1.11) involves two scaling parameters for Hamiltonian dynamics. One can assume more sophisticated cases that reveal the so-called ''strange'' or fractional kinetics [45,15] which is induced by dynamical chaos.

ACKNOWLEDGMENTS

We would like to express our gratitude to Y. Pesin for valuable discussions, and to M. Edelman for the help in preparing the figures. This work was supported by the U.S. Department of Navy, Grant No. N00014-93-1-0218, and the U.S. Department of Energy, Grant No. DE-FG02-92ER54184.

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